AP Calculus AB Exam

Virtual Student Study Session

Section 1: Area and Volume

Section 2: Particle Motion

Section 3: Accumulation Functions and Definite Integrals in Applied Contexts (Rate In/Rate Out Problems)

Detailed Solutions

Area and Volume

Area Between Two Curves

- Sketch the region and determine the points of intersection.
- Draw a small strip either as dx or dy slicing.
- Use the following templates to set up a definite integral:

  dx slicing: $A = \int_{left\,x}^{right\,x} \left(y_{top} - y_{bottom} \right) dx$ where y_{top} and y_{bottom} are written in terms of *x*.

  dy slicing: $A = \int_{bottom\,y}^{top\,y} \left(x_{right} - x_{left} \right) dy$ where x_{right} and x_{left} are written in terms of *y*.

Volume of a Solid with a Known Cross-Section

- Sketch the region and draw a representative rectangle to be used in determining whether setting up with respect to dx or dy.
- Determine the slicing direction then find the volume of the slice which will be the area of the "face" times the "thickness".
- Determine the total volume by summing up the slices using a definite integral.
- Use the following templates to set up a definite integral.

  dx *slicing:* $V = \int_{left\,x}^{right\,x} A(x)\,dx$ where *A(x)* is the area of the face written in terms of *x*.

  dy slicing: $V = \int_{bottom\,y}^{top\,y} A(y)\,dy$ *where A(y)* is the area of the face written in terms of *y*.

Volume of a Solid of Revolution

- Sketch the region to be revolved and a representative rectangle whose width can be used to determine whether integrating with dx or dy.
- Set up a definite integral after determining whether the slicing uses dx or dy so that the slicing is perpendicular to the axis of revolution.
- Identify the outside radius and the inside radius and use the appropriate template from below:

- *dx slicing:* $V = \pi \int_{left\,x}^{right\,x} \left(\left(outside\,radius \right)^2 - \left(inside\,radius \right)^2 \right) dx$

  where the outside and inside radii are written in terms of *x*.

  dy slicing: $V = \pi \int_{bottom\,y}^{top\,y} \left(\left(outside\,radius \right)^2 - \left(inside\,radius \right)^2 \right) dy$

  where the outside and inside radii are written in terms of *y*.

Multiple Choice

Area and Volume Solutions

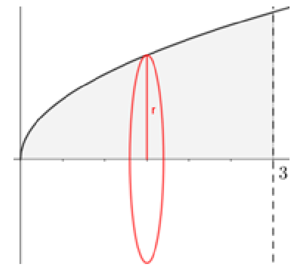
1. (calculator not allowed)

The region enclosed by the x-axis, the line $x = 3$, and the curve $y = \sqrt{x}$ is rotated about the x-axis. What is the volume of the solid generated?

(A) 3π **(B)** $2\sqrt{3}\pi$ **(C)** $\frac{9}{2}\pi$ **(D)** 9π **(E)** $\frac{36\sqrt{3}}{5}\pi$

$$volume = \int_{\text{left } x}^{\text{right } x} \underbrace{\pi r^2}_{\text{area}} \underbrace{dx}_{\text{thickness}} = \int_0^3 \pi\left(\sqrt{x}\right)^2 dx = \int_0^3 \pi x\, dx = \pi\left[\frac{1}{2}x^2\right]_0^3$$

$$= \pi\left[\frac{1}{2}(3)^2 - \frac{1}{2}(0)^2\right] = \frac{9}{2}\pi$$



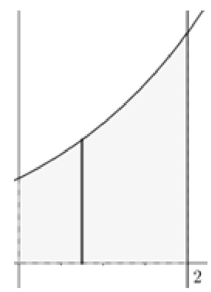
2. (calculator not allowed)

The area of the region bounded by the lines $x = 0$, $x = 2$, and $y = 0$ and the curve $y = e^{\frac{x}{2}}$ is

(A) $\frac{e-1}{2}$ **(B)** $e-1$ **(C)** $2(e-1)$ **(D)** $2e-1$ **(E)** $2e$

$$A = \int_{\text{left } x}^{\text{right } x} \underbrace{\left(y_{top} - y_{bottom}\right)}_{\text{height}} \underbrace{dx}_{\text{width}} = \int_0^2 \left(e^{\frac{x}{2}} - 0\right) dx = \int_0^2 \left(e^{\frac{x}{2}}\right) dx = \left[2e^{\frac{x}{2}}\right]_0^2$$

$$= 2\left(e^1 - e^0\right) = 2(e-1)$$

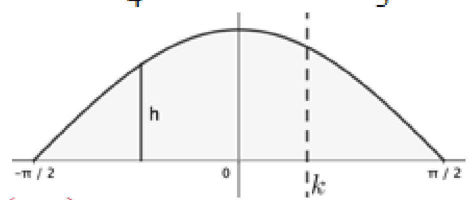


3. (calculator not allowed)

The region bounded by the x-axis and the part of the graph of $y = \cos x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is separated into two regions by the line $x = k$. If the area of the region for $-\frac{\pi}{2} \le x \le k$ is three times the area of the region for $k \le x \le \frac{\pi}{2}$, then $k =$

(A) $\arcsin\left(\frac{1}{4}\right)$ **(B)** $\arcsin\left(\frac{1}{3}\right)$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{\pi}{4}$ **(E)** $\frac{\pi}{3}$

$$A_{-\frac{\pi}{2}\le x\le k} = \int_{\text{left } x}^{\text{right } x} \underbrace{\left(y_{top} - y_{bottom}\right)}_{height} \underbrace{dx}_{width} = \int_{-\pi/2}^{k} \left(\cos(x) - 0\right) dx$$



$$= \int_{-\pi/2}^{k} \cos(x)\,dx = \left[\sin(x)\right]_{-\pi/2}^{k} = \sin(k) - \sin\left(-\frac{\pi}{2}\right) = \sin(k) - (-1) = \sin(k) + 1$$

$$A_{k\le x\le\frac{\pi}{2}} = \int_{k}^{\pi/2} \cos(x)\,dx = \left[\sin(x)\right]_{k}^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(k) = 1 - \sin(k)$$

$$A_{-\frac{\pi}{2}\le x\le k} = 3A_{k\le x\le\frac{\pi}{2}} \Rightarrow \quad \sin(k) + 1 = 3\left(1 - \sin(k)\right)$$

$$\sin(k) + 1 = 3 - 3\sin(k)$$

$$4\sin(k) = 2 \Rightarrow \sin(k) = \frac{1}{2} \Rightarrow k = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

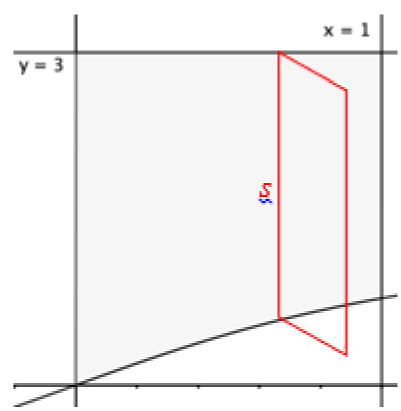
4. (calculator allowed)

The base of a solid is the region in the first quadrant bounded by the y-axis, the graph of $y = \tan^{-1} x$, the horizontal line $y = 3$, and the vertical line $x = 1$. For this solid, each cross section perpendicular to the x-*axis* is a square. What is the volume of the solid?

(A) 2.561 **(B)** 6.612 **(C)** 8.046 **(D)** 8.755 **(E)** 20.773

$$V = \int_{\text{left } x}^{\text{right } x} A(x)\,dx = \int_0^1 \underbrace{(s)^2}_{area} \underbrace{dx}_{thickness} = \int_0^1 \left(y_{top} - y_{bottom}\right)^2 dx$$

$$= \int_0^1 \left(3 - \tan^{-1} x\right)^2 dx = 6.6123\ldots$$



Free Response **Area and Volume Solutions**

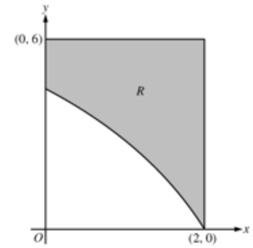
5. (calculator allowed) (2010 Form B AB/BC 1)

In the figure above, R, is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3-x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.

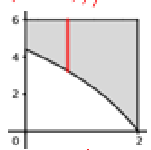
(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.



1: Correct limits in an integral in (a), (b), or (c)

(a) $A_R = \int_{\text{left } x}^{\text{right } x} \underbrace{(y_{top} - y_{bottom})}_{\text{height}} \underbrace{dx}_{\text{width}} = \int_0^2 \left(6 - \left(4\ln(3-x)\right)\right) dx$

$= 6.8166\ldots$



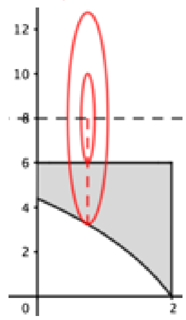
$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(b) $V = \int_{\text{left } x}^{\text{right } x} \left(\underbrace{\pi(\text{outside radius})^2}_{\text{area of large disk}} - \underbrace{\pi(\text{inside radius})^2}_{\text{area of small disk}} \right) \underbrace{dx}_{\text{thickness}}$

$= \int_0^2 \left(\pi\left(y_{top} - y_{bottom}\right)^2 - \pi\left(y_{top} - y_{bottom}\right)^2 \right) dx$

$= \int_0^2 \left(\pi\left(8 - 4\ln(3-x)\right)^2 - \pi(8-6)^2 \right) dx$

$= 168.1795\ldots$



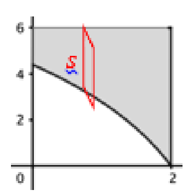
$3: \begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

Note: Form of integrand must be a difference of squares or 0/3

(c) $V = \int_{\text{left } x}^{\text{right } x} A(x)\,dx = \int_0^2 \underbrace{(s)^2}_{\text{area}} \underbrace{dx}_{\text{thickness}} = \int_0^2 \left(y_{top} - y_{bottom}\right)^2 dx$

$= \int_0^2 \left(6 - 4\ln(3-x)\right)^2 \, dx$

$= 26.2660\ldots$



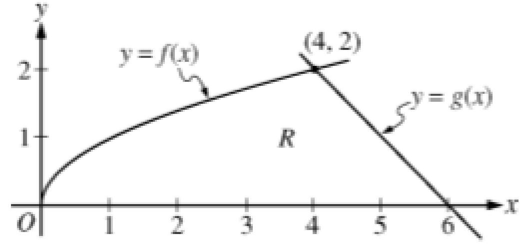
$3: \begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

Note: Form of integrand must be a difference of squares or 0/3

$3: \begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

Note: Form of integrand must be a square or 0/3

6. (calculator not allowed) (2011 Form B AB/BC 3)



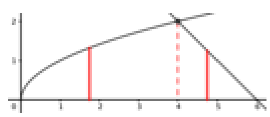
The functions f and g are given by $f(x)=\sqrt{x}$ and $g(x)=6-x$. Let R be the region bounded by the x-axis and the graphs of f and g, as shown in the figure above.

(a) Find the area of R.

(b) The region R is the base of a solid. For each y, where $0\le y\le 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(c) There is a point P on the graph of f at which the line tangent to the graph of f is perpendicular to the graph of g. Find the coordinates of point.

(a) $A_R=\int_{\text{left } x}^{\text{right } x}\underbrace{\left(y_{top}-y_{bottom}\right)}_{height}\underbrace{dx}_{width}$



$=\int_0^4\underbrace{\left(y_{top}-y_{bottom}\right)}_{height}\underbrace{dx}_{width}+\int_4^6\underbrace{\left(y_{top}-y_{bottom}\right)}_{height}\underbrace{dx}_{width}$

$=\int_0^4\left(\sqrt{x}-0\right)dx+\int_4^6\left(6-x\right)dx=\int_0^4\sqrt{x}\,dx+\underbrace{\int_4^6\left(6-x\right)dx}_{\text{area of a triangle}}$

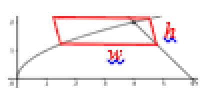
$=\left[\frac{2}{3}x^{3/2}\right]_0^4+\left[\frac{1}{2}(2)(2)\right]=\left[\frac{2}{3}(4)^{3/2}-0\right]+\left[2\right]$

$=\frac{2}{3}(8)+2=\frac{22}{3}$

3: $\begin{cases}1: \text{integral}\\ 1: \text{antiderivative}\\ 1: \text{answer}\end{cases}$

(b) $y=\sqrt{x}\Rightarrow x=y^2 \qquad y=6-x\Rightarrow x=6-y$

$V=\int_{\text{bottom } y}^{\text{top } y}A(y)\underbrace{dy}_{thickness}=\int_0^2\underbrace{h\cdot w}_{area}\underbrace{dy}_{thickness}$

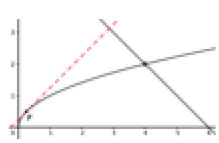


$=\int_0^2\left(\left(x_{right}-x_{left}\right)(h)\right)dy=\int_0^2\left(\left(\left(6-y\right)-y^2\right)\left(2y\right)\right)dy$

$=\int_0^2\left(2y\left(6-y-y^2\right)\right)dy$

3: $\begin{cases}2: \text{integrand}\\ 1: \text{answer}\end{cases}$

(c) $g'(x)=-1\Rightarrow\perp\text{ slope }=1$



$f'(x)=\frac{1}{2\sqrt{x}}=1\Rightarrow\sqrt{x}=\frac{1}{2}\Rightarrow x=\frac{1}{4}\Rightarrow P:\left(\frac{1}{4},\frac{1}{2}\right)$

3: $\begin{cases}1: f'(x)\\ 1: f'(x)=1\\ 1: \text{answer}\end{cases}$

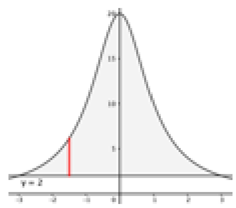
7. (calculator allowed) (2007 ABBC 1)
Let R be the region in the first and second quadrants bounded above by the graph of $y=\dfrac{20}{1+x^2}$ and below by the horizontal line $y=2$.

(a) Find the area of R.
(b) Find the volume generated when R is rotated about the x-axis.
(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

(a) $\dfrac{20}{1+x^2}=2 \Rightarrow 1+x^2=10 \Rightarrow x^2=9 \Rightarrow x=\pm 3$

$$A_R=\int_{\text{left } x}^{\text{right } x}\underbrace{\left(y_{top}-y_{bottom}\right)}_{\text{height}}\underbrace{dx}_{\text{width}}$$

$$=\int_{-3}^{3}\left(\frac{20}{1+x^2}-2\right)dx=37.9618\ldots$$



1: Correct limits in an integral in (a), (b), or (c)

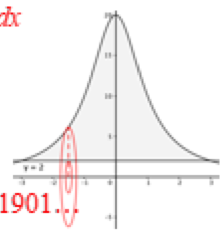
2: $\begin{cases}1:\text{integrand}\\1:\text{answer}\end{cases}$

(b) $$V=\int_{\text{left } x}^{\text{right } x}\left(\underbrace{\pi\left(\text{outside radius}\right)^2}_{\text{area of large disk}}-\underbrace{\pi\left(\text{inside radius}\right)^2}_{\text{area of small disk}}\right)\underbrace{dx}_{\text{thickness}}$$

$$=\int_{-3}^{3}\left(\pi\left(y_{top}-y_{bottom}\right)^2-\pi\left(y_{top}-y_{bottom}\right)^2\right)dx$$

$$=\int_{-3}^{3}\left(\pi\left(\frac{20}{1+x^2}-0\right)^2-\pi\left(2-0\right)^2\right)dx$$

$$=\int_{-3}^{3}\left(\pi\left(\frac{20}{1+x^2}\right)^2-\pi\left(2\right)^2\right)dx=1871.1901\ldots$$



3: $\begin{cases}2:\text{integrand}\\1:\text{answer}\end{cases}$

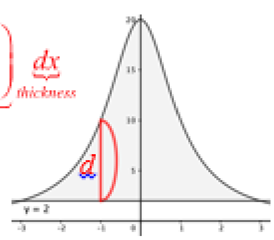
Note: Form of integrand must be a difference of squares or 0/3

(c) $$V=\int_{\text{left } x}^{\text{right } x}A(x)\underbrace{dx}_{\text{thickness}}=\int_{-3}^{3}\underbrace{\left(\frac{1}{2}\pi r^2\right)}_{\text{area}}\underbrace{dx}_{\text{thickness}}$$

$$=\int_{-3}^{3}\left(\frac{1}{2}\pi\left(\frac{d}{2}\right)^2\right)dx=\frac{1}{8}\pi\int_{-3}^{3}\left(d^2\right)dx$$

$$=\frac{1}{8}\pi\int_{-3}^{3}\left(y_{top}-y_{bottom}\right)^2dx=\frac{1}{8}\pi\int_{-3}^{3}\left(\frac{20}{1+x^2}-2\right)^2dx$$

$$=174.2684\ldots$$



3: $\begin{cases}2:\text{integrand}\\1:\text{answer}\end{cases}$

Note: Form of integrand must be a square or 0/3

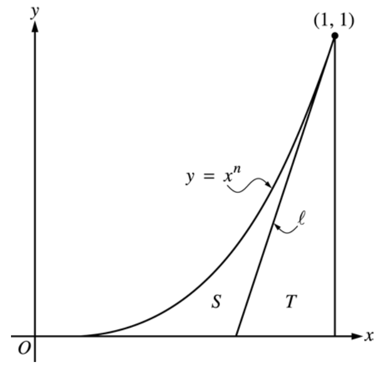
8. (calculator not allowed) (2004 Form B AB/BC 6)

Let ℓ be the line tangent to the graph of $y = x^n$ at the point (1, 1), where $n > 1$, as shown above.

(a) Find $\int_0^1 x^n \, dx$ in terms of n.

(b) Let T be the triangular region bounded by ℓ, the x-axis, and the line $x = 1$. Show that the area of T is $\frac{1}{2n}$.

(c) Let S be the region bounded by the graph of $y = x^n$, the line ℓ, and the x-axis. Express the area of S in terms of n and determine the value of n that maximizes the area of S.



(a) $\int_0^1 x^n \, dx = \left[\frac{1}{n+1}x^{n+1}\right]_0^1 = \left[\frac{1}{n+1}1^{n+1}\right] - \left[\frac{1}{n+1}0^{n+1}\right]$

$= \frac{1}{n+1}$

$2: \begin{cases} 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

(b) Let b be the length of the base of triangle T. The slope of ℓ is $\frac{1}{b} \Rightarrow \frac{dy}{dx} = nx^{n-1} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = n(1)^{n-1} = n$, the slope of the tangent line $\ell \Rightarrow \frac{1}{b} = n \Rightarrow b = \frac{1}{n}$

$A_T = \frac{1}{2}\left(\frac{1}{n}\right)(1) = \frac{1}{2n}$

$3: \begin{cases} 1: \text{slope of } \ell \text{ is } n \\ 1: \text{base of } T \text{ is } \frac{1}{n} \\ 1: \text{shows area is } \frac{1}{2n} \end{cases}$

(c) $A_S = \int_0^1 x^n \, dx - A_T = \frac{1}{n+1} - \frac{1}{2n}$

$A_S'(n) = -1(n+1)^{-2} - \left(-1(2n)^{-2}(2)\right)$

$= -\frac{1}{(n+1)^2} + \frac{2}{(2n)^2} = 0$

$\frac{1}{(n+1)^2} = \frac{2}{(2n)^2} \Rightarrow 4n^2 = 2(n+1)^2 \Rightarrow$

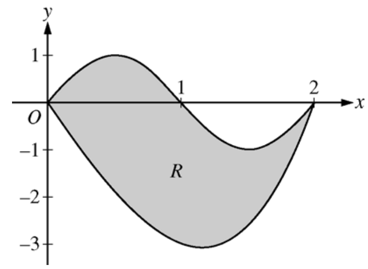
$2n^2 = (n+1)^2 \Rightarrow \sqrt{2}n = n+1 \Rightarrow \left(\sqrt{2}-1\right)n = 1$

$n = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$ maximizes the area.

$A_S'(n)$ changes from $+$ to $-$ at $n = \sqrt{2}+1$, the only critical value. A_S': $+ - - - + - - - - -$ (1, $\sqrt{2}+1$)

$4: \begin{cases} 1: \text{area of } S \text{ in terms of } n \\ 1: \frac{d}{dn}A_S \\ 1: \text{critical value of } n \\ 1: \text{reasoning} \end{cases}$

9. (calculator allowed) (2008 AB1/BC1)
Let R be the region bounded by the graphs of $y = \sin(px)$ and $y = x^3 - 4x$, as shown in the figure above.



(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

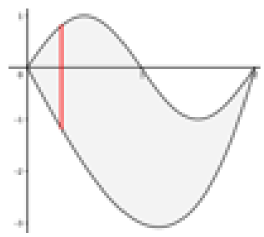
(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $x^3 - 4x = \sin(\pi x) \Rightarrow x = 0, 2$

$$A_R = \int_{\text{left } x}^{\text{right } x} \underbrace{\left(y_{top} - y_{bottom}\right)}_{\text{height}} \underbrace{dx}_{\text{width}}$$

$$= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$$

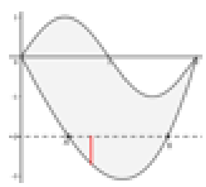


$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(b) $x^3 - 4x = -2 \Rightarrow x = 0.5391\ldots = A, 1.6751\ldots = B$

$$A = \int_{\text{left } x}^{\text{right } x} \underbrace{\left(y_{top} - y_{bottom}\right)}_{\text{height}} \underbrace{dx}_{\text{width}}$$

$$= \int_A^B \left(-2 - \left(x^3 - 4x\right)\right) dx$$

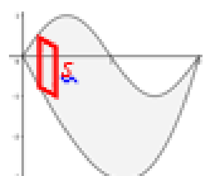


$2: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \end{cases}$

(c) $V = \int_{\text{left } x}^{\text{right } x} A(x)\,dx = \int_0^2 \underbrace{(s)^2}_{\text{area}} \underbrace{dx}_{\text{thickness}}$

$$= \int_0^2 \left(y_{top} - y_{bottom}\right)^2 dx = \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx$$

$= 9.9783\ldots$

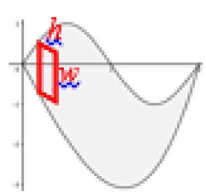


$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(d) $V = \int_{\text{left } x}^{\text{right } x} A(x)\,dx = \int_0^2 \underbrace{h \cdot w}_{\text{area}} \underbrace{dx}_{\text{thickness}}$

$$= \int_0^2 \underbrace{(3 - x)\left(y_{top} - y_{bottom}\right)}_{\text{area}} \underbrace{dx}_{\text{thickness}}$$

$$= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.3699\ldots$$



$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

Particle Motion

Typically, if a particle is moving along the x-axis at any time, t, $x(t)$ represents the position of the particle; along the y-axis, $y(t)$ is often used; along another straight line, $s(t)$ is often used. In addition, $v(t)$ is typically used to represent the velocity of the particle. In these types of particle motion problems,

- "Initially" means when time $t = 0$.
- "At the origin" means $x(t) = 0$.
- "At rest" means velocity $v(t) = 0$.
- If the velocity of the particle is positive, then the particle is moving to the right.
- If the velocity of the particle is negative, then the particle is moving to the left.
- If the acceleration of the particle is positive, then the velocity is increasing.
- If the acceleration of the particle is negative, then the velocity is decreasing.
- Speed is the absolute value of velocity.
- If the velocity and acceleration have the same sign (both positive or both negative), then speed is increasing.
- If the velocity and acceleration are opposite in sign (one is positive and the other is negative), then speed is decreasing.
- To determine total distance traveled over a time interval, you must calculate the sum of the absolute values of the differences in position between all resting points or calculate the area under the absolute value of the velocity curve, $\int_{t_1}^{t_2} |v(t)|\, dt$.
- Displacement can be determined using $displacement = \int_{t_1}^{t_2} v(t)\, dt$
- To determine the final position of a particle after motion use $Final\ Position = s(t_1) + \int_{t_1}^{t_2} v(t)\, dt$; likewise to determine the final velocity use $Final\ Velocity = v(t_1) + \int_{t_1}^{t_2} a(t)\, dt$.

Multiple Choice

Particle Motion Solutions

1. (calculator not allowed)

A particle moves along the x-axis with velocity given by $v(t) = 3t^2 + 6t$ for time $t \ge 0$. If the particle is at position $x = 2$ at time $t = 0$, what is the position of the particle at time $t = 1$?

(A) 4 **(B)** 6 (circled) **(C)** 0 **(D)** 11 **(E)** 12

$$\textit{Final Position} = s(t_1) + \int_{t_1}^{t_2} v(t)\,dt = x(0) + \int_0^1 v(t)\,dt = 2 + \int_0^1 (3t^2 + 6t)\,dt$$

$$= 2 + \left[t^3 + 3t^2\right]_0^1 = 2 + \left[1^3 + 3(1)^2\right] - \left[0^3 + 3(0)^2\right] = 6$$

2. (calculator not allowed)

A particle travels in a straight line with a constant acceleration of 3 meters per second per second. If the velocity of the particle is 10 meters per second at time 2 seconds, how far does the particle travel during the time interval when its velocity increases from 4 meters per second to 10 meters per second?

(A) 20 m **(B)** 14 m (circled) **(C)** 7 m **(D)** 6 m **(E)** 3 m

$v(t) = \int a(t)\,dt = \int 3\,dt = 3t + C$ $\quad v(2) = 10 \Rightarrow 3(2) + C = 10 \Rightarrow C = 4$

$v(t) = 3t + 4 = 4 \Rightarrow t = 0$ $\quad$ distance $= \int_0^2 |v(t)|\,dt$ $\quad$ $v(t) > 0$ on the interval $0 \le t \le 2$

$$\text{distance} = \int_0^2 |v(t)|\,dt = \int_0^2 (3t + 4)\,dt = \left[\frac{3}{2}t^2 + 4t\right]_0^2 = \left[\frac{3}{2}(2)^2 + 4(2)\right] - \left[\frac{3}{2}(0)^2 + 4(0)\right] = 6 + 8 = 14$$

3. (calculator not allowed)

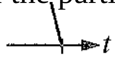
If the position of a particle on the x-axis at time t is $-5t^2$, then the average velocity of the particle for $0 \le t \le 3$ is

(A) -45 **(B)** -30 **(C)** -15 (circled) **(D)** -10 **(E)** -5

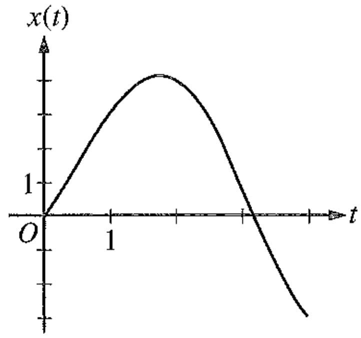
$$\text{average velocity} = \frac{x(3) - x(0)}{3 - 0} = \frac{\left(-5(3)^2\right) - \left(-5(0)^2\right)}{3} = \frac{-45 - (0)}{3} = -15$$

4. (calculator not allowed)

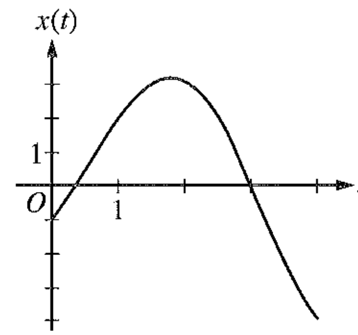
The table below gives selected values of the velocity, $v(t)$, of a particle moving along the x-axis. At time $t = 0$ the particle is at the origin. Which of the following could be the graph of the position, $x(t)$, of the particle for $0 \le t \le 4$?



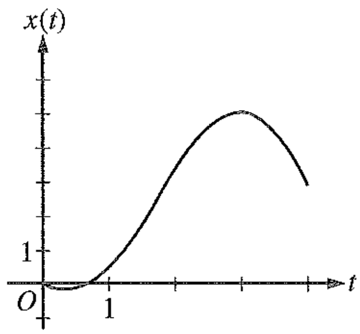
(A)



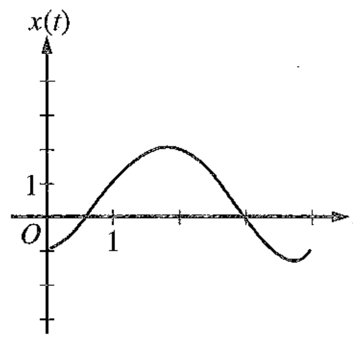
(B)



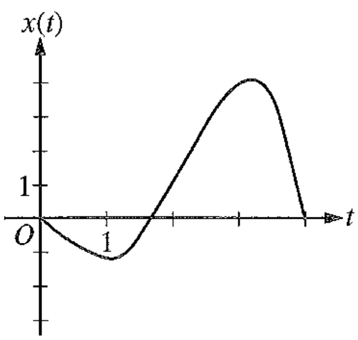
(C)



(D)



(E)



$v(0) = -1 \Rightarrow\ x(t)$ is decreasing at $x = 0$

$\Rightarrow$ eliminate (A), (B), and (D)

$v(1) = 2 \Rightarrow\ x(t)$ is increasing at $x = 1$

$\Rightarrow$ eliminate (E)

$v(3) = 0 \Rightarrow\ x(t)$ has an horizontal tangent at $x = 3$

$v(4) = -4 \Rightarrow\ x(t)$ is decreasing at $x = 4$

5. (calculator allowed)

A particle moves along the x-axis so that at any time $t \ge 0$, its velocity is given by $v(t) = 3 + 4.1\cos(0.9t)$. What is the acceleration of the particle at time $t = 4$?

(A) -2.016 **(B)** -0.677 **(C)** 1.633 **(D)** 1.814 **(E)** 2.978

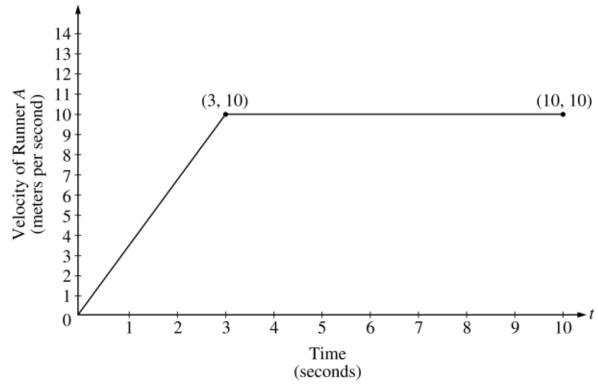
$a(4) = v'(4) = 1.6329\ldots$

Free Response

Particle Motion Solutions

6. (calculator allowed)

Two runners, *A* and *B*, run on a straight track for $0 \le t \le 10$ seconds. The graph below, which consists of two line segments, shows the velocity, in meters per second, of Runner *A*. The velocity, in meters per second, of Runner *B* is given by the function *v* defined by the function, $v(t) = \dfrac{24t}{2t+3}$.



a.) Find the velocity of Runner *A* and the velocity of Runner *B* at time *t* = 2. Indicate units of measure.

b.) Find the acceleration of Runner *A* and the acceleration of Runner *B* at time *t* = 2 seconds. Indicate units of measure.

c.) Find the total distance run by Runner *A* and the total distance run by Runner *B* over the interval $0 \le t \le 10$ seconds. Indicate units of measure.

---

a.) Runner $A: v_A(2) = \frac{10}{3}(2) = \frac{20}{3} = 6.6666\ldots$ m/s

Runner $B: v_B(2) = \frac{24(2)}{2(2)+3} = \frac{48}{7} = 6.8571\ldots$ m/s

$2: \begin{cases} 1: \text{Runner A velocity} \\ 1: \text{Runner B velocity} \end{cases}$

b.) Runner $A: a_A(2) = v_A'(2) = \frac{10}{3} = 3.3333\ldots \text{m/s}^2$

Runner $B: v_B'(2) = 1.4693\ldots \text{m/s}^2$

$2: \begin{cases} 1: \text{Runner A acceleration} \\ 1: \text{Runner B acceleration} \end{cases}$

c.) Runner A:

$\text{distance}_A = \int_0^{10} v_A(t)\,dt = \frac{1}{2}(3)(10) + (7)(10) = 85$ m

Runner B: $\text{distance}_B = \int_0^{10} v_B(t)\,dt = 83.3361\ldots$ m

$4: \begin{cases} 2: \text{Runner A distance} \\ \quad 1: \text{method} \\ \quad 1: \text{answer} \\ 1: \text{Runner B acceleration} \\ \quad 1: \text{integral} \\ \quad 1: \text{answer} \end{cases}$

1: units

7. (calculator allowed)

A particle moves along the y-axis with velocity given by $v(t) = t\sin\left(t^2\right)$ for $t \geq 0$.

a.) In which direction (up or down) is the particle moving at time $t = 1.5$? Justify your reasoning.

b.) Find the acceleration of the particle at time $t = 1.5$. Is the velocity of the particle increasing at time $t = 1.5$? Justify your reasoning.

c.) Given that $y(t)$ is the position of the particle at time t and that $y(0) = 3$, find $y(2)$.

d.) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

Solution	Scoring
a.) $v(1.5) = 1.1671\ldots \Rightarrow$ up because $v(1.5) > 0$	$2: \begin{cases} 1: \text{considers } v(1.5) \\ 1: \text{conclusion and reason} \end{cases}$
b.) $a(1.5) = v'(1.5) = -2.0487\ldots$ velocity is not increasing because $a(1.5) < 0$	$2: \begin{cases} 1: a(1.5) \\ 1: \text{conclusion and reason} \end{cases}$
c.) $\textit{Final Position} = s(t_1) + \int_{t_1}^{t_2} v(t)\,dt$ $y(2) = y(0) + \int_0^2 v(t)\,dt = 3 + \int_0^2 v(t)\,dt = 3.8268\ldots$	$3: \begin{cases} 1: \text{definite integral of } v(t) \\ 1: \text{limits and constant} \\ 1: \text{answer} \end{cases}$
d.) total distance $= \int_0^2 \left\lvert v(t) \right\rvert dt = 1.1731\ldots$	$2: \begin{cases} 1: \text{definite integral of } \left\lvert v(t) \right\rvert \\ 1: \text{answer} \end{cases}$

Free Response

Using Accumulation and Defnite Integral in Applied Contexts (Rate In/Rate Out Problem)

1. (calculator allowed)

A popular 24-hour health club, Get Swole, has 29 people using its facility at time $t = 0$. During the time interval $0 \le t \le 20$ hours, people are entering the health club at the rate

$$E(t) = -0.018t^2 + 11 \text{ people per hour.}$$

During the same time period people are leaving the health club at the rate of

$$L(t) = 0.013t^2 - 0.25t + 8 \text{ people per hour.}$$

a.) Is the number of people in the facility increasing or decreasing at time $t = 11$? Explain your reasoning.

b.) To the nearest whole number, how many people are in the health club at time $t = 20$.

c.) At what time t, for $0 \le t \le 20$, is the amount of people in the health club a maximum? Justify your answer.

d.) If the health club has less than 10 people using its facility, the owner will serve everyone a special health drink. Set up, but do not solve, an inequality involving one or more integrals that could be used to find the time, H, when the number of people in the health club is less that or equal to 10.

---

a.) $E(11) = 8.822$ $\quad$ $L(11) = 6.823$

The number of people in the facility is increasing because the people are entering at a rate (8.822 people per hour) higher than the rate people are leaving (6.823 people per hour) at $t = 11$ hours.

$2:\begin{cases} 1: \text{considers } E(11) \text{ and } L(11) \\ 1: \text{answer with reason} \end{cases}$

b.) $P(t)$ is the number of people in the facility at time t

$$P(20) = P(0) + \int_0^{20} E(t)\,dt - \int_0^{20} L(t)\,dt = 29 + (172 - 144.666...)$$

$$= 56.3333\ldots \approx 56 \text{ people}$$

$2:\begin{cases} 1: \text{integral(s)} \\ 1: \text{answer} \end{cases}$

c.) $P'(t) = 0 \Rightarrow E(t) = L(t) \Rightarrow t = 14.66397 = a$

t	$P(t)$
0	29
a	67.2876…
20	56.3333…

The number of people is maximum at $t = 14.66397$ hours

$3:\begin{cases} 1: \text{sets } E(t) - L(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

d.) $P(H) == 29 + \int_0^H [E(t) - L(t)]\,dt \le 10$

$2:\begin{cases} 1: \text{integral(s)} \\ 1: \text{inequality} \end{cases}$

2. (calculator allowed)

A parking garage has 230 cars in it when it opens at 8 AM ($t = 0$). On the interval $0 \le t \le 10$, cars enter the parking garage at a rate modeled by the function

$$E(t) = 58\cos\left(0.163t - 0.642\right)$$

and leave the parking garage at a rate modeled by the function

$$L(t) = 65\cos\left(0.281t\right) + 7.1$$

beginning at 9 AM and continuing until 6 PM ($t = 10$). Both $E(t)$ and $L(t)$ are measured in cars per hour while t is measured in hours.

a.) How many cars enter the parking garage over the interval $t = 0$ to $t = 10$ hours?

b.) Find $E'(5)$. Using correct units, explain the meaning of this value in the context of the problem.

c.) Find the number of cars in the parking garage at time $t = 10$. Show the work that leads to your answer.

d.) Find the time t on $0 \le t \le 10$ when the number of cars in the parking garage is a maximum. To the nearest whole number, what is the maximum number of cars in the parking garage? Justify your answer.

---

a.) $\int_0^{10} E(t)\,dt = 510.1599\ldots$ cars enter the parking garage

$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

b.) $E'(5) = -1.6273\ldots$ The number of cars entering the parking garage is changing at a rate of $-1.6273\ldots$ cars per hour at time $t = 5$ hours.

$2: \begin{cases} 1: E'(5) \\ 1: \text{meaning} \end{cases}$

c.) $230 + \int_0^{10} E(t)\,dt - \int_1^{10} L(t)\,dt = 665.103\ldots\text{cars}$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

d.) $E(t) - L(t) \ne 0$ on $0 \le t \le 10$

t	cars
0	230
10	1016.6699…

At $t = 10$ hours the number of cars is at a maximum of 1017 cars

$3: \begin{cases} 1: \text{sets } E(t) - L(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$