Unit 3.5 Procedures for Calculating Derivatives

Topic Question 1

Which of the following could be used to find the slope of the line tangent to the curve
$\tan^{-1}(x-2y+2)=x^2-3y+\tan^{-1}(2)-1$?

Ⓐ $\frac{1}{1+(x-2y+2)^2}=2x-3\frac{dy}{dx}$

Ⓑ $\frac{-1}{1+(x-2y+2)^2}=2x-3$

Ⓒ $\frac{1-2\frac{dy}{dx}}{1+(x-2y+2)^2}=2x-3\frac{dy}{dx}$

Ⓓ $\frac{1-2\frac{dy}{dx}}{1+(x-2y+2)}=2x-3\frac{dy}{dx}$

$\underbrace{\tan^{-1}(x-2y+2)}_{\text{chain}}=x^2-3y+\tan^{-1}(2)-1$?

$\frac{1}{1+(x-2y+2)^2}\left(1-2\frac{dy}{dx}\right)=2x-3\frac{dy}{dx}+0-0$

Topic Question 2

x	3	6
$f(x)$	4	5
$f'(x)$	$\frac{1}{3}$	$\frac{3}{4}$

INVERSE EXISTS

Continuous INVERSE EXISTS

The table above gives selected values for a differentiable and increasing function f and its derivative. Let g be the increasing function given by $g(x)=f(x)+f(2x)$, where $g(3)=f(3)+f(6)=9$. Which of the following describes a correct process for finding $(g^{-1})'(9)$?

Ⓐ $(g^{-1})'(9)=\frac{1}{g'(g^{-1}(9))}=\frac{1}{g'(3)}$ and $g'(3)=f'(3)+2f'(6)$

Ⓑ $(g^{-1})'(9)=\frac{1}{g'(g^{-1}(9))}=\frac{1}{g'(3)}$ and $g'(3)=f'(3)+f'(6)$

Ⓒ $(g^{-1})'(9)=g'(g^{-1}(9))=g'(3)$ and $g'(3)=f'(3)+f'(6)$

Ⓓ $(g^{-1})'(9)=g'(g^{-1}(9))=g'(3)$ and $g'(3)=f'(3)+2f'(6)$

chain

$g'(x)=f'+\overbrace{f'(2x)\cdot 2}$

$g'(3)=f'(3)+2f'(2\cdot 3)$

g: $(3,9)$, $m=g'(3)$

g^{-1}: $(9,3)$, $m=\frac{1}{g'(3)}$

Unit 3.5 Procedures for Calculating Derivatives

Topic Question 3

Which of the following does not require the use of the chain rule to find $\frac{dy}{dx}$?

(A) $y = \sin^{-1}(3x^2 - 4)$ Chain

(B) $3x^6 - 4y^2 = 2xy^5 + 7$ implicit Chain

Chain Chain

(C) $y = 3x^2 - \sqrt{x} + \frac{2}{x}$ Explicit

(D) $\cos(x + y) + 2^y - x = 0$ Chain

Chain Chain Chain

Topic Question 6

Which of the following could be used to find the slope of the line tangent to the curve $\sin^{-1}(2x^2 + y^2) = \frac{2}{x} + y^2$?

(A) $\frac{1}{\sqrt{1-(2x^2+y^2)^2}} = \frac{-2}{x^2} + 2y\frac{dy}{dx}$

(B) $\frac{4x+2y}{\sqrt{1-(2x^2+y^2)^2}} = \frac{-2}{x^2} + 2y$

(C) $\frac{4x+2y\frac{dy}{dx}}{\sqrt{1-(2x^2+y^2)^2}} = \frac{-2}{x^2} + 2y\frac{dy}{dx}$

(D) $\frac{4x+2y\frac{dy}{dx}}{\sqrt{1-(2x^2+y^2)}} = \frac{-2}{x^2} + 2y\frac{dy}{dx}$

$\sin^{-1}(2x^2 + y^2) = \frac{2}{x} + y^2$

chain chain chain

$\left(\frac{2}{x}\right)' = (2x^{-1})'$
$= -2x^{-2}$
$= \frac{-2}{x^2}$

$\frac{1}{\sqrt{1 - 2x^2+y^2}}\left(4x + 2y\frac{dy}{dx}\right) = \frac{-2}{x^2} + 2y\frac{dy}{dx}$

Unit 3.5 Procedures for Calculating Derivatives

In exercises 1 – 6, find the derivative of each of the following functions.

1. $f(x) = \left(\dfrac{x+5}{x^2+2}\right)^3$

Chain Rule

Quotient Rule

$$\frac{df}{dx} = 3\left(\frac{x+5}{x^2+2}\right)^2 \cdot \frac{(1)(x^2+2) - (x+5)(2x)}{(x^2+2)^2}$$

$$= \frac{3(x+5)^2}{(x^2+2)^2} \cdot \frac{x^2+2-2x^2-10x}{(x^2+2)^2}$$

$$= \frac{3(x+5)^2(-x^2-10x+2)}{(x^2+2)^4}$$

$$\frac{df}{dx} = \frac{-3(x+5)^2(x^2+10x-2)}{(x^2+2)^4}$$

2. $f(x) = \sqrt{\dfrac{2x+3}{x-2}} = \left(\dfrac{2x+3}{x-2}\right)^{\frac{1}{2}}$

Chain Rule

Quotient Rule

$$\dot{f} = \frac{1}{2}\left(\frac{2x+3}{x-2}\right)^{-\frac{1}{2}} \cdot \frac{2(x-2)-(2x+3)(1)}{(x-2)^2}$$

$$= \frac{1}{2}\left(\frac{x-2}{2x+3}\right)^{\frac{1}{2}} \cdot \frac{2x-4-2x-3}{(x-2)^2}$$

$$= \frac{(x-2)^{\frac{1}{2}}}{2(2x+3)^{\frac{1}{2}}} \cdot \frac{-7}{(x-2)^2}$$

$$= \frac{-7}{2(2x+3)^{\frac{1}{2}}(x-2)^{3/2}}$$

$$\dot{f} = \frac{-7}{2\sqrt{2x+3}\sqrt{(x-2)^3}}$$

3. $f(x) = 3x^2\cos(2x)$

Chain Rule

$$f'(x) = 6x\cdot\cos(2x) + 3x^2(-\sin(2x))\cdot 2$$

Product Rule

$$f'(x) = 6x\cos(2x) - 6x^2\sin(2x)$$

4. $p(x) = \dfrac{\cot x}{\sin x}$

Quotient Rule

$$\dot{p}(x) = \frac{(-\csc^2 x)\cdot\sin x - \cot x(\cos x)}{(\sin x)^2}$$

$$= \frac{-\frac{1}{\sin^2 x}\cdot\sin x - \frac{\cos x}{\sin x}\cdot\cos x}{\sin^2 x}$$

$$= \frac{-\frac{1}{\sin x} - \frac{\cos^2 x}{\sin x}}{\sin^2 x}$$

$$= \frac{\frac{-1-\cos^2 x}{\sin x}}{\sin^2 x}$$

$$= \frac{-1-\cos^2 x}{\sin^3 x}$$

$$= \frac{-1-(1-\sin^2 x)}{\sin^3 x}$$

$$= \frac{-1-1+\sin^2 x}{\sin^3 x}$$

$$\dot{p}(x) = \frac{-2+\sin^2 x}{\sin^3 x}$$ } many correct versions of this answer

Unit 3.5 Procedures for Calculating Derivatives

~~For exercises 7 and 8,~~ find the value of the derivative of the function at the given point.

6. $f(\theta) = \sin(2\theta)\cos(2\theta)$ when $\theta = \frac{\pi}{4}$

Product Rule

SoT

$f'(\theta) = \underbrace{\cos(2\theta)\cdot 2}_{\text{Chain}} \cdot \cos(2\theta) + \sin(2\theta)\underbrace{(-\sin(2\theta))\cdot 2}_{\text{Chain}}$

$f'(\theta) = 2\cos^2(2\theta) - 2\sin^2(2\theta)$

$f'(\frac{\pi}{4}) = 2\cos^2(2\cdot\frac{\pi}{4}) - 2\sin^2(2\cdot\frac{\pi}{4})$

$= 2\cos^2(\frac{\pi}{2}) - 2\sin^2(\frac{\pi}{2})$

$= 2(0)^2 - 2(1)^2$

$= 0 - 2(1)$

$f'(\frac{\pi}{4}) = -2$

① ②

7. Find the following limit. Explain the reasoning that you used to arrive at your answer.

$$\lim_{h\to 0}\frac{\cos 3(x+h) - \cos(3x)}{h}$$

② The definition of derivative is $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ so $f(x) = \cos 3x$.

① $f'(x) = -\sin(3x)\cdot 3$

$f'(x) = -3\sin(3x)$

Use the table below to complete exercises 13 – 14.

x	*f*(*x*)	*f* ′(*x*)	*g*(*x*)	*g*′(*x*)
–2	1	–1	2	4
–1	3	–2	1	1
0	–1	2	–2	–3

8. If $H(x)=\sqrt{f(x)\cdot g(x)}$, is the graph of *H*(*x*) increasing or decreasing when *x* = –1? Give a reason for your answer.

$$H(x) = [f(x)\cdot g(x)]^{\frac{1}{2}}$$

Chain

Product

$$\frac{dH}{dx} = \frac{1}{2}[f\cdot g]^{-\frac{1}{2}}\cdot(f'g+fg')$$

$$\frac{dH}{dx} = \frac{f'(x)g(x)+f(x)g'(x)}{2\sqrt{f(x)g(x)}}$$

$$\left.\frac{dH}{dx}\right|_{x=-1} = \frac{f'(-1)g(-1)+f(-1)g'(-1)}{2\sqrt{f(-1)g(-1)}}$$

$$= \frac{(-2)\cdot(1)+(3)(1)}{2\sqrt{(3)(1)}}$$

$$= \frac{-2+3}{2\sqrt{3}}$$

$$\left.\frac{dH}{dx}\right|_{x=-1} = \frac{1}{2\sqrt{3}}$$

H(x) is increasing when x=-1 because H'(x) > 0 at x=-1

9. If $P(x)=(2f(x)+g(x))^{\frac{2}{3}}$, what is the value of $P'(0)$?

CHAIN

$$P'(x) = \frac{2}{3}[2f(x)+g(x)]^{-1/3}\cdot[2f'(x)+g'(x)]$$

$$P'(x) = \frac{2[2f'(x)+g'(x)]}{3\sqrt[3]{2f(x)+g(x)}}$$

$$P'(x) = \frac{4f'(x)+2g'(x)}{3\sqrt[3]{2f(x)+g(x)}}$$

$$P'(0) = \frac{4f'(0)+2g'(0)]}{3\sqrt[3]{2f(0)+g(0)}}$$

$$= \frac{4(2)+2(-3)]}{3\sqrt[3]{2\cdot(-1)+(-2)}}$$

$$= \frac{8-6}{3\sqrt[3]{-2-2}}$$

$$= \frac{2}{3\sqrt[3]{-4}}$$

$$P'(0) = \frac{-2}{3\sqrt[3]{4}}$$

15. Find the $\boxed{\text{equation of the normal line}}$ to the graph of $h(x)=\tan(3x)$ when $x=\frac{\pi}{12}$.

PoT $\left(\frac{\pi}{12}, 1\right)$

$$h\left(\frac{\pi}{12}\right) = \tan\left(3\cdot\frac{\pi}{12}\right)$$

$$= \tan\left(\frac{\pi}{4}\right)$$

$$h\left(\frac{\pi}{12}\right) = 1$$

SoT: m = 6

Chain

$$h'(x) = 3\sec^2(3x)$$

$$h'\left(\frac{\pi}{12}\right) = 3\sec^2\left(3\cdot\frac{\pi}{12}\right)$$

$$= 3\sec^2\left(\frac{\pi}{4}\right)$$

$$= 3\left(\sqrt{2}\right)^2$$

$$= 3\cdot 2$$

$$h'\left(\frac{\pi}{12}\right) = 6$$

SoN: $m = -\frac{1}{6}$

$$y-1 = -\frac{1}{6}\left(x-\frac{\pi}{12}\right)$$